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the locus of a point, such that tangents being drawn from it to two equal intersecting circles, their rectangle may be constant, and equal to the square of half the common chord.

If the circles do not intersect, and if the rectangle under the tangents be equal to the square of the tangent to either from the middle point between the centres, the locus will give the curve derived from an hyperbola, whose real axis is greater than the imaginary.

To get the curve when the imaginary axis is greater than the real, we must take the locus of a point, such that lines being drawn from it to meet the circumferences of two equal, non-intersecting circles, and subtending right angles at their centres, their rectangle may be constant, and equal to the sum of the squares of the radius, and of half the distance between the centres.

There exist in spherical geometry numerous properties of an analogous character.

The Rev. Charles Graves made a communication relative to the new functions employed by him in the interpretation of his theory of Algebraic Triplets.

The three functions A, M, N, defined in p. 60, possess so many interesting properties that they seem to deserve distinctive appellations. The first is symmetrical with respect to its two amplitudes, ϕ and χ; and its properties are in the main analogous to those of the trigonometrical function *cosine*. On the other hand, M and N are not symmetrical functions of ϕ and χ; but either of them may be obtained from the other by interchanging the two amplitudes; and they correspond in many respects to the trigonometrical *sine*. Mr. Graves proposes then to call A the *cotresine*, and M and N the *tresines* of ϕ and χ: and he designates them respectively by the symbols cotr $[\phi, \chi]$, tres $[\phi, \chi]$, tres $[\chi, \phi]$.

As in trigonometry $\cos\phi = \cos(2i\pi + \phi)$, and $\sin\phi =$

$\sin(2i\pi + \phi)$, i being any integer number; so also in this calculus,

$$\begin{aligned} \mathrm{cotr}\,[\phi, \chi] &= \mathrm{cotr}\,[\phi + 2i\tau,\ \chi - 2i\tau] \\ \mathrm{tres}\,[\phi, \chi] &= \mathrm{tres}\,[\phi + 2i\tau,\ \chi - 2i\tau] \end{aligned}$$

where $\tau = \frac{\pi}{\sqrt{3}}$. The quantity τ holding in the present theory the same place that π does in the calculus of sines. From the theorem of Moivre we learn that

$$(\cos\phi + \sqrt{-1}.\sin\phi)^m = \cos m(\phi + 2i\pi) + \sqrt{-1}.\sin m\,(\phi + 2i\pi).$$

The corresponding theorem already announced by Mr. Graves may be written as follows:

$$\begin{gathered} (\mathrm{cotr}[\phi, \chi] + \iota\,\mathrm{tres}[\phi, \chi] + \iota^2\mathrm{tres}[\chi, \phi])^m \\ = \mathrm{cotr}\,[m(\phi + 2i\tau), m(\chi - 2i\tau)] + \iota\,\mathrm{tres}\,[m(\phi + 2i\tau),\ m(\chi - 2i\tau)] \\ + \iota^2\mathrm{tres}\,[m(\chi - 2i\tau),\ m(\phi + 2i\tau)]. \end{gathered}$$

Among the most important consequences flowing from the preceding theorem is a mode of resolving an equation of the form

$$z^{3n} - pz^{2n} + qz^n - r = 0$$

into its cubic factors. For this purpose we must first reduce the equation to the form

$$y^{3n} - 3\,\mathrm{cotr}(\mathrm{A}, \mathrm{B})\,y^{2n} + 3\,\mathrm{cotr}(-\mathrm{A}, -\mathrm{B})\,y^n - 1 = 0. \qquad (b)$$

Now it will be found that if we multiply together the three expressions

$$\begin{aligned} &x - \mathrm{cotr}\,[\phi, \chi] - \mathrm{tres}[\phi, \chi] - \mathrm{tres}\,[\chi, \phi] \\ &x - \mathrm{cotr}\,[\phi, \chi] - a\,\mathrm{tres}\,[\phi, \chi] - a^2\mathrm{tres}\,[\chi, \phi] \\ &x - \mathrm{cotr}\,[\phi, \chi] - a^2\mathrm{tres}[\phi, \chi] - a\,\mathrm{tres}\,[\chi, \phi] \end{aligned}$$

in which a stands for $\frac{-1 \pm \sqrt{-3}}{2}$, the cube root of $+1$, the product will be

$$x^3 - 3\,\mathrm{cotr}[\phi, \chi]\,x^2 + 3\,\mathrm{cotr}\,[-\phi,\ -\chi]\,x - 1.$$

Hence it may be proved that each of the cubic factors of (b) will be of the form

$$y^3 - 3\,\mathrm{cotr}\left[\frac{1}{n}(\mathrm{A}+2i\tau),\ \frac{1}{n}(\mathrm{B}-2i\tau)\right]y^2$$
$$+3\,\mathrm{cotr}\left[-\frac{1}{n}(\mathrm{A}+2i\tau),\ -\frac{1}{n}(\mathrm{B}-2i\tau)\right]y - 1.$$

From the result just obtained Mr. Graves deduces a singular geometrical theorem analogous to the celebrated one of Cotes.

But it must be observed here that the geometrical propositions to which we are led in interpreting formulæ involving tresines, do not, in general, relate to the *lengths* of lines. If x, y, z and x', y', z' be respectively the rectangular coordinates of two points in space, the distance between them is expressed by the quantity $\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$. But this is not the function of x, y, z, x', y', z' that the formulæ in question commonly bring before us: they most frequently introduce, instead of it, the function

$$\sqrt[3]{(x'-x)^3+(y'-y)^3+(z'-z)^3-3(x'-x)(y'-y)(z'-z)}$$

which Mr. Graves proposes to call the *cubic modulus* of the line joining the points x, y, z and x', y', z'. This cubic modulus may be written in a form which suggests important consequences.

If we put

$$\begin{aligned} x &= m\,\mathrm{cotr}[\phi,\chi] & x' &= m'\,\mathrm{cotr}[\phi',\chi'] \\ y &= m\,\mathrm{tres}[\phi,\chi] & y' &= m'\,\mathrm{tres}[\phi',\chi'] \\ z &= m\,\mathrm{tres}[\chi,\phi] & z' &= m'\,\mathrm{tres}[\chi',\phi'] \end{aligned}$$

it becomes

$$\sqrt[3]{m'^3-3m'^2m\,\mathrm{cotr}[\phi-\phi',\chi-\chi']+3m'm^2\,\mathrm{cotr}[\phi'-\phi,\chi'-\chi]-m^3}$$

Thus we have the modulus of the line joining two points in space expressed by means of the differences of their corresponding amplitudes and the moduli of the right lines drawn to them from the origin :—a result analogous to the fundamental proposition in plane trigonometry, by which the length of one side of a triangle is found from the two remaining sides and the included angle. Having now got the notion of the modulus of a right line in space, we may easily advance to the conception of the modulus of a curved line: for we may regard it as the sum of the moduli of the elements of the curve: so that to find m, the modulus of the curve itself, we employ the formula

$$m = \int \sqrt[3]{dx^3 + dy^3 + dx^3 - 3\,dx\,dy\,dz}.$$

The right line drawn from the point x, y, z to another point, z', y', z', may be looked upon as having amplitudes as well as a modulus. In order to determine them we put

$$x'-x = m\,\mathrm{cotr}[\phi, \chi] : y'-y = m\,\mathrm{tres}[\phi, \chi] : z'-z = m\,\mathrm{tres}[\chi, \phi]:$$

m being the modulus of the right line.

For the purpose of illustrating his views, Mr. Graves has discussed the surface whose equation in rectangular coordinates is

$$x^3 + y^3 + z^3 - 3xyz = 1.$$

As might be expected, this surface possesses numerous properties which admit of being stated in such a manner as to exhibit a striking similarity to those of the circle. The three coordinates x, y, z belonging to any point on it, may be put equal to $\mathrm{cotr}[\phi, \chi]$, $\mathrm{tres}[\phi, \chi]$, and $\mathrm{tres}[\chi, \phi]$; and we may call ϕ and χ the amplitudes of the point. It will also be convenient to designate the point, whose amplitudes are $-\phi$ and $-\chi$, as *reciprocal* to the point whose amplitudes are ϕ and χ.

Amongst other theorems relating to this surface, which Mr. Graves proposes to name *the surface of tresines*, the following may be considered as deserving attention:

1. *The angle between the tangent planes at any two points on the surface is equal to that between the radii drawn to the two reciprocal points.*

2. *The measure of curvature at any point is equal to* $-r_1^{-4}$: r_1 *denoting the radius of the reciprocal point.*

3. *The surface is one of revolution: and it is its own polar reciprocal with reference to a sphere whose centre is at the origin of coordinates and whose radius is unity.*

The curve of double curvature, or *curve of tresines,* defined by the equations

$$x = \mathrm{cotr}\,[\phi, o]$$
$$y = \mathrm{tres}\,[\phi, o]$$
$$z = \mathrm{tres}\,[o, \phi]$$

appears likewise to be most fertile in properties analogous to those of the circle. The following were stated by Mr. Graves:

1. *The angle between the tangents at any two points on the curve is equal to that between the two corresponding radii.*

2. *The angle between the osculating planes at any two points on the curve is equal to the angle between the radii drawn to the two reciprocal points.*

3. *The angle of contact at a point on the curve is therefore equal to the torsion at the reciprocal point.*

4. *The element of the curve described by the reciprocal point is double the elementary area described by the radius vector.*

5. *The polar reciprocal of the developable surface formed by the osculating planes is itself* the curve of tresines.

6. *The product of the radii of curvature at any point and at its reciprocal is equal to unity.*

7. *The radius vector traces a logarithmic spiral upon a plane parallel to the symmetric plane.*

Mr. John Neville read a paper on the maximum Amount of Resistance required to sustain Banks of Earth and other Materials.

Let CDE be any bank of earth, sand, or other material, and CE the position of the line of repose with respect to the horizon and bank; then if we put CA, the perpendicular from C on DE produced $= h$, the $\angle$ DEA $= b$; the angle ECA $= c$; and the $\angle$ ACF, which the fracture CF makes